Asymptotic expansion for cycles in homology classes for graphs

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Abstract. In this paper we give an asymptotic expansion including error terms for the number of cycles in homology classes for connected graphs. Mainly, we obtain formulae about the coefficients of error terms which depend on the homology classes and give two examples of how to calculate the coefficient of first error term.

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1. Introduction. To estimate the number of closed orbits for certain flows has been studied by many authors such as [**2**], [**4**] and [**9**]. The error terms of asymptotic expansion were not known until the works of Dolgopyat on Anosov flows, where he obtained strong results on the contractivity of transfer operator. These results led Anantharaman [**1**] , Pollicott and Sharp [**10**] and Liu [**5**] to find full expansions of expression for the number of closed orbits for Anosov flows. The key to these methods lies in reduction of calculating closed orbits of an Anosov flow to calculating closed orbits of a suspended flow or to calculating periodic points of a subshift of finite type [**8**].

This strategy led us to consider the number of cycles of a connected graph in this article.

A graph G is defined to be a pair (V, E), where V is a set $\{V_1, V_2, \ldots, V_n\}$ of elements called vertices, and E is a family $(e_1, e_2, \ldots, e_m)$ of (undirected) edges joining elements of V. There may be more than one edge joining the two vertices. If a vertex is joined to itself by a edge, we call this edge a loop. We will only consider the connected finite graphs in this article.

It is convenient to speak of graph in which each edge has an orientation attached to it. In this case, we call the graph an oriented graph. We can associate to an undirected graph G with n vertices and m edges , an oriented graph G_o with n vertices and $2m$ edges. An oriented graph G_o is a pair $(V, \mathbb{E})$, where $\mathbb{E}$ is a set of ordered pairs of elements of V. For $e \in \mathbb{E}$, we denote by $I(e)$ the initial endpoint of e and $T(e)$ the terminal endpoint of e.

We label the edges of oriented graph G_o by $1, 2, \ldots, 2m$. For example, Figure 1.1 is a undirected graph with 3 vertices and 4 edges, Figure 1.2 is

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the corresponding oriented graph to Figure 1.1.

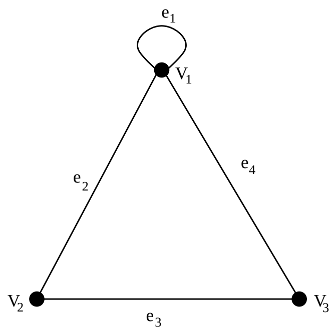

Figure 1.1

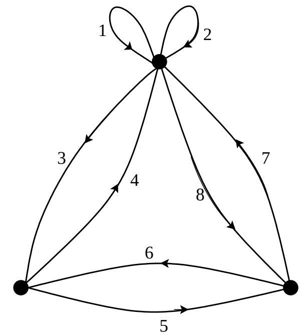

Figure 1.2

A chain is a sequence $u = (u_1, u_2, \ldots, u_q)$ of edges of G_o such that each edge in the sequence has one endpoint in common with its predecessor in the sequence and its other endpoint in common with its successor in the sequence, i.e., $T(u_i) = I(u_{i+1})$, $i = 1, 2, \ldots, q-1$.

A cycle γ is a chain such that the two endpoints of the chain are the same vertex, i.e., a chain $(u_1, u_2, \ldots, u_q)$ is a cycle if $T(u_q) = I(u_1)$. The edge length of a cycle γ is defined by the number of edges in γ. We say a cycle $\gamma = (u_1 \ldots, u_q)$ has backtracking if $u_i = -u_{i+1}$ for some i, $1 \leq i \leq q-1$, where $-u_{i+1}$ is the reverse of u_{i+1}.

We assign a length to each edge and denote the length of e_i by $l(e_i)$. For the corresponding oriented graph, we have $l(e_i) = l(-e_i)$. The length of a chain $(u_1, u_2, \ldots, u_n)$ is $l(u_1) + l(u_2) + \cdots + l(u_n)$.

We denote by $H_1(G, \mathbb{Z})$ the homology group of G. For convenience, we assume that $H_1(G, \mathbb{Z}) = \mathbb{Z}^b$. Otherwise, we can write $H_1(G, \mathbb{Z}) = \mathbb{Z}^b \oplus H$. Since the torsion subgroup H is finite, the results then will only differ by a multiplicative constant.

Let Γ be the set of cycles in graph G. For $\gamma \in \Gamma$ we denote by $[\gamma]$ the homology class in $H_1(G, \mathbb{Z})$. Let $l(\gamma)$ be the length of γ.

For $\alpha \in H_1(G, \mathbb{Z})$, let

$$\pi(T, \alpha) = \#\{\gamma \in \Gamma, l(\gamma) \leq T, [\gamma] = \alpha\}.$$

We will give the asymptotic formulae for $\pi(T, \alpha)$ which is similar to the case of homologically full transitive Anosov flow [**5**]. But we will concentrate on how to calculate the first error term for special cases in this paper.

We briefly outline the contents of this article. In section 2, we explain how, through the use of symbolic dynamics, the counting problem for cycles can be reduced to one for periodic points for a subshift of finite type. In section 4, we introduce a function $Z(s, v)$ and derive some important properties of its analytic extension which be used to obtain the formula for

distribution of cycles including error terms, that is Theorem 1. We specify the coefficient for the first error term in this section. In the last two sections we will give two examples for how to calculate the coefficient of the first error term, where we use two different methods. Since the calculating of coefficient of first error term involve the derivatives of a function $\beta(u)$ which be introduced in section 3, we will give some formulae of derivatives of $\beta(u)$ in this section.

2. Symbolic dynamics. For a graph G with m edges there exists a $2m \times 2m$ matrix A_G with zero-one entries associated with the corresponding oriented graph G_o. The matrix A_G can be defined by following. For $1 \leq i, j \leq 2m$, if the terminal endpoint of edge i is equal to the initial endpoint of edge j then $A(i,j) = 1$, otherwise $A(i,j) = 0$. For example, the matrix associated with Figure 1.2 is

$$A_G = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Let $A = (a_{ij})$ be a $k \times k$ matrix, we say A is non-negative if $a_{ij} \geq 0$ for all i, j. Such a matrix is called irreducible if for any pair i, j there is some n such that $a_{i,j}^{(n)} > 0$ where $a_{ij}^{(n)}$ is (i,j)-th element of A^n, i.e., $a_{ij}^{(n)} = A_{ij}^n$. The matrix A is aperiodic if there exists $n > 0$ such that $a_{ij}^{(n)} > 0$ for all i, j.

It is easy to see that the graph G is connected if and only if associated A_G is irreducible. So if A_G is aperiodic then G is connected. But if G is connected, A_G may be not aperiodic, for example, a bipartite graph is connected but the associated matrix is not aperiodic. Where we say a graph G is bipartite if its vertex set can be partitioned into two classes such that no two adjacent vertices belong to the same class. A graph is bipartite if and only if it possesses no cycles of odd edge length.
However, It is easy to prove that

Lemma 1. *If G is connected and it is not bipartite, then associated matrix A_G is aperiodic.*

We will only consider connected graph G whose corresponding matrix A_G is aperiodic.

We define Σ_A by

$$\Sigma_A = \left\{ x \in \prod_0^\infty \{1, 2, \ldots, 2m\} : A_G(x_i, x_{i+1}) = 1, \forall i \in \mathbb{Z}^+ \right\}.$$

The subshift of finite type $\sigma : \Sigma_A \to \Sigma_A$ is defined by subshift

$$(\sigma x)_i = x_{i+1}.$$

We define $r : \Sigma_A \to \mathbb{R}^+$ by $r(x) = l(x_0)$, then

$$l(x_0, x_1, \ldots, x_{n-1}) = r(x) + r(\sigma x) + \cdots + r(\sigma^{n-1}x) =: r^n(x).$$

There is a one-one correspondence between closed orbits $\{x, \sigma x, \ldots, \sigma^{n-1}x\}$ for $\sigma : \Sigma_A \to \Sigma_A$ and cycles of the graph G. The least length of corresponding cycle is $r^n(x)$.

There exists $f = (f_1, f_2, \ldots, f_b) : \Sigma_A \to \mathbb{R}^b$ such that for $\gamma \in \Gamma$, $[\gamma] = f^n(x)$ for some n, x with $\sigma^n x = x$. We can even make $f(x)$ just depend on one co-ordinate, i.e., $f(x) = f(x_0)$ [**8**].

Remark:
If A_G is connected but not aperiodic then it is a bipartite graph. In this case, we can decompose Σ_A by $\Sigma_A = \Sigma_0 \bigcup \Sigma_1$ satisfies

$$\sigma : \Sigma_0 \longrightarrow \Sigma_1, \qquad \Sigma_1 \longrightarrow \Sigma_0.$$

So $\sigma^2 : \Sigma_A \to \Sigma_A$ satisfies $\sigma^2 : \Sigma_0 \to \Sigma_0, \qquad \Sigma_1 \to \Sigma_1$. we define

$$R(x) := r^2(x) = r(x) + r(\sigma x)$$

and

$$F(x) := f^2(x) = f(x) + f(\sigma x).$$

There is one-one correspondence between closed orbits $\{x, (\sigma^2)x, \ldots, (\sigma^2)^{n-1}x\}$ for $\sigma^2 : \Sigma_0 \to \Sigma_0$ or $(\Sigma_1 \to \Sigma_1)$ and cycles of the graph G. The least length of corresponding cycle is $R^n(x)$, which is same as A_G is aperiodic.

In order to obtain a positive result we shall only consider the graphs satisfying the following conditions [**10**].

(A) *Weak-Mixing.* The closed subgroup of $\mathbb{R}$ generated by $\{l(\gamma)\}(\gamma \in \Gamma)$ is $\mathbb{R}$.
(B) *Approximability Condition.* There exist three cycles γ_1, γ_2 and γ_3 with least lengths $l(\gamma_1)$, $l(\gamma_2)$ and $l(\gamma_3)$, respectively, such that

$$\xi = \frac{l(\gamma_1) - l(\gamma_2)}{l(\gamma_2) - l(\gamma_3)}$$

is badly approximable, i,e., there exists $\alpha > 0$ and $C > 0$ such that we have $|\xi - \frac{p}{q}| \geq \frac{C}{q^\alpha}$, for all $p, q \in \mathbb{Z}(q > 0)$.

The set of ξ satisfying this condition is a large set. For example, it is a set of full measure in the real line. Moreover, its complement has Hausdorff dimension zero.

3. Derivatives of function $\beta(u)$. In this section, we first briefly review the pressure function then calculate the derivatives of associated function $\beta(u)$. The pressure function $P : C(\Sigma_A) \to \mathbb{R}$ is defined by

$$P(g) = \sup_{m \in M_\sigma} \{h_m(\sigma) + \int g dm\},$$

where M_σ is the set of σ-invariant probability measures and $h_m(\sigma)$ is the entropy of σ with respect to m. Let h be the unique number such that $P(-hr) = 0$. There is no loss in generality in assuming $\int f d\mu_{-hr} = 0$, where μ_{-hr} is the equilibrium state of $-hr$.

For $u \in \mathbb{R}^b$, the function $\beta(u) : \mathbb{R}^b \to \mathbb{R}$ is defined by

$$P(-\beta(u)r + \langle u, f\rangle) = 0 \tag{1}$$

Then $\beta(u)$ is an analytic function on $\mathbb{R}^b$ and $\beta(u)$ is strictly convex in each u_i, $i = 1, 2, \ldots, b$, where $\langle u, f\rangle = \sum_{i=1}^b u_i f_i$. Now we can extend $\beta(u)$ to complex values of the argument. For all $u \in \mathbb{R}^b$, $P(-\beta(u)r + \langle u, f\rangle) = 0$ and $P(-sr + \langle u + iv, f\rangle)$ is analytic for $(s, u + iv)$ in a neighbourhood of $\mathbb{R} \times \mathbb{R}^b$ in $\mathbb{C} \times \mathbb{C}^b$. Since

$$\left[\frac{\partial P(-sr + \langle u, f\rangle)}{\partial s}\right]_{s=s_0} = -\int r d\mu_{-s_0 r + \langle u, f\rangle} \neq 0,$$

by the implicit function theorem, $\beta(u)$ can extend to an analytic function on a neighbourhood of $\mathbb{R}^b$ in $\mathbb{C}^b$ by the equation

$$P(-\beta(u + iv)r + \langle u + iv, f\rangle) = 0.$$

We have $\beta(0) = h$, since $P(-hr) = 0$.

When estimating $\pi(T, \alpha)$, the formulae which arise involve derivatives of the function $\beta(u)$. In this section, we shall calculate these derivatives up to the fourth order. Partial differentiating (1) with respect to u_i yields

$$\frac{\partial P}{\partial \beta}\frac{\partial \beta}{\partial u_i} + \frac{\partial P}{\partial u_i} = 0. \tag{2}$$

Since

$$\left[\frac{\partial P(-\beta(u)r)}{\partial \beta}\right]_{\beta=\beta(0)=h} = -\int r d\mu_{-hr},$$

and

$$\left[\frac{\partial P(-hr+<u,f>)}{\partial u_i}\right]_{u=0}=\int f_i d\mu_{-hr},$$

we have

$$\frac{\partial\beta(0)}{\partial u_i}=\frac{\int f_i d\mu_{-hr}}{\int r d\mu_{-hr}}=0.$$

For obtaining expression of $\nabla\beta^2(0)$, partial differentiate (2) with respect to u_j, and note $\nabla\beta(0)=0$ we have

$$\frac{\partial^2\beta(0)}{\partial u_i\partial u_j}=\frac{1}{\int r d\mu_{-hr}}\left[\frac{\partial^2 P(-hr+<u,f>)}{\partial u_i\partial u_j}\right]_{u=0}.$$

There is another expression for $\partial^2\beta(0)/\partial u_i\partial u_j$, that is

$$\frac{\partial^2\beta(0)}{\partial u_i\partial u_j}=\frac{1}{\int r d\mu_{-hr}}\lim_{n\to\infty}\frac{1}{n}\int f_i^n f_j^n d\mu_{-hr}.$$

We refer to [**4**] for this formula.

The third and fourth order derivatives with respect to u_i are following.

$$\frac{\partial^3\beta(0)}{\partial u_i\partial u_j\partial u_m}=\frac{1}{\int r d\mu_{-hr}}\left\{\left[\frac{\partial^3 P(-hr+<u,f>)}{\partial u_i\partial u_j\partial u_m}\right]_{u=0}\right.$$
$$\left.+\left[\left(\frac{\partial^2 P}{\partial\beta\partial u_i}\frac{\partial^2\beta}{\partial u_j\partial u_m}+\frac{\partial^2 P}{\partial\beta\partial u_j}\frac{\partial^2\beta}{\partial u_i\partial u_m}+\frac{\partial^2 P}{\partial\beta\partial u_m}\frac{\partial^2\beta}{\partial u_i\partial u_j}\right)(-\beta r+<u,f>)\right]_{\beta=h,u=0}\right\}.$$

$$\frac{\partial^4\beta(0)}{\partial u_i\partial u_j\partial u_m\partial u_n}=\frac{1}{\int r d\mu_{-hr}}\left\{\left[\frac{\partial^4 P(-hr+<u,f>)}{\partial u_i\partial u_j\partial u_m\partial u_n}\right]_{u=0}\right.$$
$$+\left[\frac{\partial^2 P}{\partial\beta^2}\left(\frac{\partial^2\beta}{\partial u_i\partial u_j}\frac{\partial^2\beta}{\partial u_m\partial u_n}+\frac{\partial^2\beta}{\partial u_i\partial u_m}\frac{\partial^2\beta}{\partial u_j\partial u_n}+\frac{\partial^2\beta}{\partial u_i\partial u_n}\frac{\partial^2\beta}{\partial u_j\partial u_m}\right)\right]$$
$$+\left(\underbrace{\frac{\partial^3 P}{\partial\beta\partial u_i\partial u_j}\frac{\partial^2\beta}{\partial u_m\partial u_n}+\cdots+\frac{\partial^3 P}{\partial\beta\partial u_m\partial u_n}\frac{\partial^2\beta}{\partial u_i\partial u_j}}_{6\quad items}\right)$$
$$\left.+\left(\underbrace{\frac{\partial^2 P}{\partial\beta\partial u_i}\frac{\partial^3\beta}{\partial u_j\partial u_m\partial u_n}+\cdots}_{4\quad items}\right)(-\beta r+<u,f>)\Bigg|_{\beta=h,u=0}\right\}.$$

For $k>4$, $\nabla\beta^k(0)$ is more complicated. But for some special graph G, $\nabla\beta^k(0)$ may be easy to calculate.

4. Distribution of cycles. Let g be of class C^∞ with compact support. For $\alpha \in H_1(G,\mathbb{Z})$, we first estimate the auxiliary function

$$\pi_g(T,\alpha) = \sum_{\gamma\in\Gamma,[\gamma]=\alpha} g(l(\gamma)-T).$$

Let $\hat{g}$ be the Fourier transform of g, By Fourier's Inverse Transform Formula,

$$\begin{aligned}\pi_g(T,\alpha) &= \sum_{\gamma\in\Gamma} \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^b/\mathbb{Z}^b} \hat{g}(-i\sigma+t)e^{-(\sigma+it)(l(\gamma)-T)}e^{\langle 2\pi iv,[\gamma]\rangle}e^{-\langle 2\pi iv,\alpha\rangle}dvdt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^b/\mathbb{Z}^b} Z(\sigma+it,v)e^{(\sigma+it)T}\hat{g}(-i\sigma+t)e^{-2\pi i<v,\alpha>}dvdt,\end{aligned}$$

where we have defined

$$Z(s,v) = Z(\sigma+it,v) = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{x\in Fix_n} e^{-(\sigma+it)r^n(x)+2\pi i<v,f^n(x)>} + A(\sigma+it,v),$$

with $Fix_n = \{x\in\Sigma_A, \sigma^n x = x\}$ and $A(\sigma+it,v)$ is analytic when $\sigma > h-\epsilon$ for some $\epsilon>0$.

It is well-known that when $Res = \sigma > \beta(0) = h$, $Z(s,v)$ is absolutely convergent. For the behaviour of $Z(s,v)$ in $Res < h$, we can determine the domain of $Z(s,v)$ by studying the norm of the transfer operator $\mathcal{L}_{s,v}$, which is detailed discussed in [**3**]. Same procedure as that in [**5**] or more originally in [**10**], we have following proposition.

Proposition 1. *Under conditions (A) and (B), there exist* $B>0$*,* $c>0$*,* $\epsilon>0$*,* $\lambda>0$*,* $\rho>0$ *and a open set* V_0*, a neighbourhood of* 0 *in* $\mathbb{R}^b/\mathbb{Z}^b$ *such that*
(1) $Z(s,v)$ *is analytic in* $\{s=\sigma+it : \sigma > h-\frac{c}{|t|^\rho}, |t|>B\}$*. And in this domain* $|Z(s,v)| = O(|t|^\lambda)$*;*
(2) $Z(s,v)+\log(s-\beta(iv))$ *is analytic in* $\{(s,v): v\in V_0, \sigma>h-\epsilon, |t|\le B\}$*;*
(3) $Z(s,v)$ *is analytic in* $\{(s,v): v\notin \bar{V_0}, \sigma>h-\epsilon, |t|\le B\}$*.*

Using Proposition 1, we can estimate

$$\pi_g(T,\alpha) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^b/\mathbb{Z}^b} Z(\sigma+it,v)e^{(\sigma+it)T}\hat{g}(-i\sigma+t)e^{-2\pi i<v,\alpha>}dvdt.$$

We divide $\mathbb{R}^b/\mathbb{Z}^b$ into V_0 and $\mathbb{R}^b/\mathbb{Z}^b - V_0$ (V_0 is a neighbourhood of 0 in $\mathbb{R}^b/\mathbb{Z}^b$ in proposition 1). For $v\in\mathbb{R}^b/\mathbb{Z}^b - V_0$, $Z(s,v)$ is analytic in $\{s = \sigma+it : \sigma>h-\epsilon, |t|<B\}\cup\{s=\sigma+it:\sigma>h-\frac{c}{|t|^\rho}, |t|>B\}$. It is easy to estimate the integral over $\mathbb{R}^b/\mathbb{Z}^b - V_0$. For $v\in V_0$, using suitable contour

integral and Residue Formula we can transfer the integral over $\sigma > h$ to integral over $\{\sigma + it : \sigma > h - \frac{c}{|t|^\rho}, |t| > B\}$. Then expanding the integral function by Taylor Formula we can estimate the integral over V_0. The details are similar to that for estimating closed orbits in homology class for Anosov flow [**5**]. We have

Theorem 1. *Let G be connected finite undirected graph.Assume that $H_1(G, \mathbb{Z}) = \mathbb{Z}^b$. If g is of class C^∞ with compact support, there exist $h > 0$ such that*

$$\pi_g(T, \alpha) = \frac{e^{Th}}{T^{b/2+1}} \left(\sum_{n=0}^{N} \frac{c_{n,g}(\alpha)}{T^n} + O\left(\frac{1}{T^{N+1}}\right) \right) \text{ as } T \to \infty, \tag{3}$$

for all $N \in \mathbb{N}$. If we write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_b)$ then the constants $c_{n,g}(\alpha) = \sum_{i_1+i_2+\cdots+i_b=0}^{2n} c_{i_1 i_2 \ldots i_b} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_b^{i_b}$, $c_{i_1 \ldots i_b}$ are constants which only depend on the lengths of edges of graph G and g.

We assume that $\rho < 1$ in Proposition 1 and let $\delta = [\frac{1}{\rho}] - 1$, same as that in [**5**] for homologically full transitive Anosov flow, the error term is not worse than $O\left(\frac{1}{T^\delta}\right)$ when we use approximation argument to estimate $\pi(T, \alpha)$. We have following theorem.

Theorem 2. *Let G be connected finite undirected graph, $H_1(G, \mathbb{Z}) = \mathbb{Z}^b$. There exist $h > 0$ and $\delta > 0$ such that*

$$\pi(T, \alpha) = \frac{e^{Th}}{T^{b/2+1}} \left(c_0 + \sum_{n=1}^{N} \frac{c_n(\alpha)}{T^n} + O\left(\frac{1}{T^\delta}\right) \right) \text{ as } T \to \infty$$

for $N = \delta - 1$, where $c_0 > 0$ is a constant. If we write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_b)$ then the constants $c_n(\alpha) = \sum_{i_1+i_2+\cdots+i_b=0}^{2n} c_{i_1 i_2 \ldots i_b} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_b^{i_b}$, $c_{i_1 \ldots i_b}$ are constants which only depend on the lengths of edges of graph G.

Analogy to the calculating closed geodesics in [**6**], the coefficient $c_{1,g}(\alpha)$ in (3) is following.

$$c_{1,g}(\alpha) = -\sum_{i,j=1}^{b} a_{ij} \alpha_i \alpha_j + \sum_{i=1}^{b} b_i \alpha_i + c_1$$

with

$$\begin{aligned}
a_{ij} &= 2\pi^2 \hat{g}(-ih) \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)} v_i v_j dv, \\
b_i &= \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)} F_1(iv) \cdot (2\pi i v) dv, \\
c_1 &= \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)} F_2(iv) dv.
\end{aligned}$$

Where

$$\begin{aligned}
F_1(iv) &= \frac{1}{6}\hat{g}(-ih)\beta^{(3)}(0)\cdot(iv)^3 + \bar{g}_0^{(1)}(0)\cdot(iv),\\
F_2(iv) &= \frac{1}{72}\hat{g}(-ih)\left[2(\beta^{(3)}(0)\cdot(iv)^3)^2 + 3\beta^{(4)}(0)\cdot(iv)^4\right]\\
&+ \frac{1}{6}\bar{g}_0^{(1)}(0)\cdot(iv)\beta^{(3)}(0)\cdot(iv)^3 + \frac{1}{2}\bar{g}_0^{(2)}(0)\cdot(iv)^2 + \bar{g}_1^{(0)}(0),
\end{aligned}$$

with $\bar{g}_j(iv) = \frac{d^j\hat{g}(-i\beta(iv))}{ds^j}$.

Since $\beta''(0)$ is positive definite, there exists a linear transformation $v = Mu$ such that $\langle v, \beta''(0)v\rangle = \sum_{k=0}^{b} u_k^2$, where $v = (v_1, v_2, \ldots, v_b)^T$, $u = (u_1, u_2, \ldots, u_b)^T$, M is a $b \times b$ matrix with $\det M > 0$. That is, there exists a matrix M such that $M^T\beta''(0)M = Id$. Hence

$$\begin{aligned}
a_{ij} &= 2\pi^2\hat{g}(-ih)\int_{\mathbb{R}^b} e^{-\frac{1}{2}\sum_{k=1}^{b}u_k^2}\left(\sum_{l=1}^{b} M_{il}u_l\right)\left(\sum_{m=1}^{b} M_{jm}u_m\right)\det M du\\
&= 2\pi^2\hat{g}(-ih)\det M\sum_{l=1}^{b} M_{il}M_{jl}\int_{\mathbb{R}^b} e^{-\frac{1}{2}\sum_{k=1}^{b}u_k^2}u_l^2 du\\
&= (2\pi)^{\frac{b}{2}+2}\frac{\hat{g}(-ih)}{2}\det M\sum_{l=1}^{b} M_{il}M_{jl}.
\end{aligned}$$

It is easy to see $b_i = 0$ in here because $\pi_g(T,\alpha) = \pi_g(T,-\alpha)$. The formula for the constant c is still complicated since we need to calculate $\beta^{(3)}(0)$ and $\beta^{(4)}(0)$.

We take g close $\chi_{[-\infty,0]}$, then $\pi_g(T,\alpha) = \pi(T,\alpha)$. Furthermore

$$\hat{g}(-is) = \int_{-\infty}^{0} e^{sy}dy = \frac{1}{s}.$$

Hence $\hat{g}(-ih) = \frac{1}{h}$ and $\bar{g}_0(-is) = \hat{g}(-is) = \frac{1}{s}$. However $\bar{g}_0(iv) = \hat{g}(-i\beta(iv)) = \frac{1}{\beta(iv)}$ and $\bar{g}_1(iv) = -\frac{1}{\beta^2(iv)}$. In this case, $\bar{g}_0^{(1)}(0) = 0$, $\bar{g}_0^{(2)}(0) = \frac{\nabla^2\beta(0)}{h^2}$ and $\bar{g}_1^{(0)}(0) = -\frac{1}{h^2}$.
So

$$a_{ij} = \frac{(2\pi)^{\frac{b}{2}+2}}{2h}detM\sum_{l=1}^{b} M_{il}M_{jl}.$$

Theorem 3. *Let G be connected finite undirected graph. There exist $h > 0$ and $\delta > 0$ that that*

$$\pi(T,\alpha) = \frac{e^{Th}}{T^{b/2+1}}\left(c_0 + \sum_{n=1}^{N}\frac{c_n(\alpha)}{T^n} + O\left(\frac{1}{T^\delta}\right)\right) \quad as\ T \to \infty$$

with

$$c_1(\alpha) = -\frac{(2\pi)^{\frac{b}{2}+2}}{2h} detM \sum_{i,j=1}^{b} \sum_{l=1}^{b} M_{il} M_{jl} \alpha_i \alpha_j + c_{1,0},$$

where $M = (M_{ij})$ *is a* $b \times b$ *matrix such that* $(MM^T)^{-1} = \beta''(0)$ *and* $c_{1,0}$ *is a constant which is independent of* α.

5. Example 1. Let us consider simple case, where G is a graph with one vertex and k edges which form k loops. In this case,

$$A_G = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$

Let the lengths of edges be $l_1, l_2, \dots, l_k$ respectively such that conditions (A) and (B) are satisfied.

We define $r : \Sigma_A \to \mathbb{R}$ by

$$r(x) = r(x_0) = \begin{cases} l_1 & \text{if } x_0 = 1 \\ l_1 & \text{if } x_0 = 2 \\ \cdots & \cdots \\ l_k & \text{if } x_0 = 2k-1 \\ l_k & \text{if } x_0 = 2k \end{cases}$$

The homology group of G, $H_1(G, \mathbb{Z}) \cong \mathbb{Z}^k$.

$f : \Sigma_A \to \mathbb{Z}^k$ is defined by $f(x) = f(x_0) = (f_1(x_0), \dots, f_k(x_0))$ such that

$$f_i(x) = f_i(x_0) = \begin{cases} 1 & \text{if } x_0 = 2i-1 \\ -1 & \text{if } x_0 = 2i \\ 0 & \text{otherwise} \end{cases}$$

In order to obtain the formula $c_1(\alpha)$, we need to calculate $\beta''(0)$ and $F_1(iv)$, $F_2(iv)$ which involve $\beta^{(3)}(0)$ and $\beta^{(4)}$. Next we compute $\beta''(0)$, $\beta^{(3)}(0)$ and $\beta^{(4)}(0)$.

Noting $P(-hr) = 0$, we have

$$e^{P(-hr+<u,f>)} = \sum_{l=1}^{2k} e^{-hr(l)+<u,f(l)>}. \tag{4}$$

In this case,

$$\int r d\mu_{-hr} = \sum_{l=1}^{2k} r(l) e^{-hr(l)} = 2 \sum_{i=1}^{k} l_i e^{-hl_i}.$$

By directly calculattion, we have

$$\left[\frac{\partial P(-hr+<u,f>)}{\partial u_i}\right]_{u=0} = \sum_{l=1}^{2k} e^{-hr(l)} f_i(l) = e^{-hl_i}[1+(-1)] = 0,$$

$$\left[\frac{\partial^2 P(-hr+<u,f>)}{\partial u_i \partial u_j}\right]_{u=0} = \begin{cases} 2e^{-hl_i} & \text{if } i=j \\ 0 & \text{if } i \neq j, \end{cases}$$

$$\left[\frac{\partial^2 P}{\partial \beta \partial u_i}(-\beta r+<u,f>)\right]_{\beta=h,u=0} = 0, \forall i,$$

$$\left[\frac{\partial^3 P(-hr+<u,f>)}{\partial u_i \partial u_j \partial u_m}\right]_{u=0} = \sum_{l=1}^{2k} f_i(l) f_j(l) f_m(l) e^{-hr(l)} = 0.$$

Useing the formulae in section 3, we have

$$\nabla \beta(0) = 0,$$

$$\frac{\partial^2 \beta(0)}{\partial u_i \partial u_j} = \begin{cases} \frac{e^{-hl_i}}{\sum_{i=1}^k l_i e^{-hl_i}} & \text{if } i=j \\ 0 & \text{if } i \neq j, \end{cases}$$

$$\frac{\partial^3 \beta(0)}{\partial u_i \partial u_j \partial u_m} = 0.$$

For calculating $\beta^{(4)}(0)$, we also need followings.

$$\left[\frac{\partial^4 P(-hr+<u,f>)}{\partial u_i \partial u_j \partial u_m \partial u_n}\right]_{u=0} = \begin{cases} -4e^{-h(l_i+l_m)} & \text{if } i=j \neq m=n \\ -4e^{-h(l_i+l_n)} & \text{if } i=m \neq j=n \\ -4e^{-h(l_i+l_j)} & \text{if } i=n \neq j=m \\ 2e^{-hl_i} - 12e^{-2hl_i} & \text{if } i=j=m=n \\ 0 & \text{otherwise,} \end{cases}$$

$$\left[\frac{\partial^2 P}{\partial \beta^2}(-\beta r+<u,f>)\right]_{\beta=h,u=0} = 2\sum_{i=1}^k l_i^2 e^{-hl_i} - 4\left(\sum_{i=1}^k l_i e^{-hl_i}\right)^2,$$

and

$$\left[\frac{\partial^3 P(-hr+<u,f>)}{\partial \beta \partial u_i \partial u_j}\right]_{u=0} = \begin{cases} -2l_i e^{-hl_i} + 4e^{-hl_i}\sum_{s=1}^k l_s e^{-hl_s} & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

These above imply that

$$\begin{aligned}
\frac{\partial^4\beta(0)}{\partial u_i^4} &= \frac{1}{2\sum_{s=1}^k l_s e^{-hl_s}}\left\{2e^{-hl_i}-12e^{-2hl_i}+\left[2\sum_{s=1}^k l_s^2 e^{-hl_s}-4\left(\sum_{s=1}^k l_s e^{-hl_s}\right)^2\right]\right.\\
&\times \left.\frac{3e^{-2hl_i}}{\left(\sum_{s=1}^k l_s e^{-hls}\right)^2}+6\left(-2l_i e^{-hl_i}+4e^{-hl_i}\sum_{s=1}^k l_s e^{-hl_s}\right)\cdot\frac{e^{-hl_i}}{\sum_{s=1}^k l_s e^{-hl_s}}\right\}\\
&= : \frac{8d_i e^{-2hl_i}}{\sum_{s=1}^k l_s e^{-hl_s}},
\end{aligned}$$

where

$$d_i=\frac{1}{16}\left\{2e^{hl_i}-\frac{12l_i}{\sum_{s=1}^k l_s e^{-hl_s}}+\frac{6\sum_{s=1}^k l_s^2 e^{-hl_s}}{(\sum_{s=1}^k l_s e^{-hl_s})^2}\right\}. \tag{5}$$

And

$$\begin{aligned}
\frac{\partial^4\beta(0)}{\partial u_i^2\partial u_j^2} &= \frac{1}{2\sum_{s=1}^k l_s e^{-hl_s}}\left\{-4e^{-h(l_i+l_j)}+\left[2\sum_{s=1}^k l_s^2 e^{-hl_s}-4\left(\sum_{s=1}^k l_s e^{-hl_s}\right)^2\right]\right.\\
&\times \frac{e^{-h(l_i+l_j)}}{\left(\sum_{s=1}^k l_s e^{-hl_s}\right)^2}+\left(-2l_i e^{-hl_i}+4e^{-hl_i}\sum_{s=1}^k l_s e^{-hl_s}\right)\cdot\frac{e^{-hl_j}}{\sum_{s=1}^k l_s e^{-hl_s}}\\
&+ \left.\left(-2l_j e^{-hl_j}+4e^{-hl_j}\sum_{s=1}^k l_s e^{-hl_s}\right)\cdot\frac{e^{-hl_i}}{\sum_{s=1}^k l_s e^{-hl_s}}\right\}\\
&= : \frac{24d_{ij}}{\sum_{s=1}^k l_s e^{-hl_s}}e^{-h(l_i+l_j)},
\end{aligned}$$

where

$$d_{ij}=\frac{1}{24}\left\{\frac{\sum_{s=1}^k l_s^2 e^{-hl_s}}{\left(\sum_{s=1}^k l_s e^{-hl_s}\right)^2}-\frac{l_i+l_j}{\sum_{s=1}^k l_s e^{-hl_s}}\right\}. \tag{6}$$

Otherwise,

$$\frac{\partial^4\beta(0)}{\partial u_i\partial u_j\partial u_m\partial u_n}=0.$$

Let $\sum_{i=1}^k l_i e^{-hl_i}=\frac{1}{c'}$. By the preceding section we have

$$a_{ij}=2\pi^2\hat{g}(-ih)\int_{\mathbb{R}^k} e^{-\frac{1}{2}c'\sum_{m=1}^k e^{-hl_m}v_m^2}v_i v_j dv.$$

So if $i \neq j$, $a_{ij} = 0$. For $i = j$,

$$\begin{aligned}
a_{ii} &= 2\pi^2 \hat{g}(-ih) \int_{\mathbb{R}^k} e^{-\frac{1}{2}c' \sum_{m=1}^k e^{-hl_m} v_m^2} v_i^2 dv \\
&= 2\pi^2 \hat{g}(-ih) \frac{\sqrt{2\pi}}{\sqrt{c' e^{-hl_1}}} \times \cdots \times \frac{e^{hl_i}\sqrt{2\pi}}{c'\sqrt{c' e^{-hl_i}}} \times \cdots \times \frac{\sqrt{2\pi}}{\sqrt{c' e^{-hl_k}}} \\
&= \frac{(2\pi)^{\frac{k}{2}+2} \hat{g}(-ih) e^{hl_i}}{2c'^{\frac{k}{2}+1}\sqrt{e^{-h(l_1+\cdots+l_k)}}}.
\end{aligned}$$

Substituting $\hat{g}(-ih) = 1/h$ and let $\xi = \frac{(2\pi)^{\frac{k}{2}+2}}{2hc'^{\frac{k}{2}+1}\sqrt{e^{-h(l_1+\cdots+l_k)}}}$,

$$a_{ij} = \begin{cases} \xi e^{hl_i} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

In order to obtain constant $c_{1,0}$, we first calculate $F_2(iv)$. Since

$$\beta^{(4)}(0) \cdot (iv)^4 = 24c' \sum_{i \neq j} d_{ij} e^{-h(l_i+l_j)} (iv_i)^2 (iv_j)^2 + 8c' \sum_{i=1}^k d_i e^{-2hl_i} (iv_i)^4,$$

we have

$$\begin{aligned}
&F_2(iv) = \frac{3\hat{g}(-ih)}{72} \\
&\times \left[24c' \sum_{i \neq j} d_{ij} e^{-h(l_i+l_j)} (iv_i)^2 (iv_j)^2 + 8c' \sum_{i=1}^k d_i e^{-2hl_i} (iv_i)^4 \right] + \frac{\bar{g}_0^{(2)}(0)}{2} \cdot (iv)^2 + \bar{g}_1^{(0)}(0) \\
&= c' \hat{g}(-ih) \sum_{i \neq j} d_{ij} e^{-h(l_i+l_j)} v_i^2 v_j^2 + \frac{c' \hat{g}(-ih)}{3} \sum_{i=1}^k d_i e^{-2hl_i} v_i^4 + \frac{\bar{g}_0^{(2)}(0)}{2} \cdot (iv)^2 + \bar{g}_1^{(0)}(0).
\end{aligned}$$

Hence

$$\begin{aligned}
&c_{1,0} = \int_{\mathbb{R}^k} e^{-\frac{1}{2}c' \sum_{i=1}^k e^{-hl_i} v_i^2} F_2(iv) dv = c' \hat{g}(-ih) \sum_{i \neq j} d_{ij} e^{-h(l_i+l_j)} \\
&\times \frac{\sqrt{2\pi}}{\sqrt{c' e^{-hl_1}}} \times \cdots \frac{e^{hl_i}\sqrt{2\pi}}{c'\sqrt{c' e^{-hl_i}}} \times \cdots \times \frac{e^{hl_j}\sqrt{2\pi}}{c'\sqrt{c' e^{-hl_j}}} \times \cdots \times \frac{\sqrt{\pi}}{\sqrt{c' e^{-hl_k}}} \\
&+ \frac{c' \hat{g}(-ih)}{3} \sum_{i=1}^k d_i e^{-2hl_i} \frac{\sqrt{2\pi}}{\sqrt{c' e^{-hl_1}}} \times \cdots \frac{3e^{2hl_i}\sqrt{2\pi}}{c'^2\sqrt{c' e^{-hl_i}}} \times \cdots \times \frac{\sqrt{2\pi}}{\sqrt{c' e^{-hl_k}}} \\
&+ \int_{\mathbb{R}^k} e^{-\frac{1}{2}c' \sum_{i=1}^k e^{-hl_i} v_i^2} \left(\frac{\bar{g}_0^{(2)}(0)}{2} \cdot (iv)^2 + \bar{g}_1^{(0)}(0) \right) dv \\
&= d_1 \hat{g}(-ih) \frac{(2\pi)^{\frac{k}{2}}}{c'^{\frac{k}{2}+1}\sqrt{e^{-h(l_1+\cdots+l_k)}}} + d_2 \hat{g}(-ih) \frac{(2\pi)^{\frac{k}{2}}}{c'^{\frac{k}{2}+1}\sqrt{e^{-h(l_1+\cdots+l_k)}}} + C \\
&= \frac{1}{2\pi^2}(d_1 + d_2)\xi + C,
\end{aligned}$$

where

$$d_1 = \sum_{i \neq j} d_{ij} \qquad d_2 = \sum_{i=1}^{k} d_i \tag{7}$$

and

$$\begin{aligned}
C &= \int_{\mathbb{R}^k} e^{-\frac{1}{2}c' \sum_{i=1}^k e^{-hl_i} v_i^2} \left(\frac{\bar{g}_0^{(2)}(0)}{2} \cdot (iv)^2 + \bar{g}_1^{(0)}(0) \right) dv \\
&= \int_{\mathbb{R}^k} e^{-\frac{1}{2}c' \sum_{i=1}^k e^{-hl_i} v_i^2} \left(-\frac{c'}{2h^2} \sum_{i=1}^{k} e^{-hl_i} v_i^2 - \frac{1}{h^2} \right) dv \\
&= -\frac{k}{2h^2} \frac{(2\pi)^{\frac{k}{2}}}{c'^{\frac{k}{2}} \sqrt{e^{-h(l_1+\cdots+l_k)}}} - \frac{1}{h^2} \frac{(2\pi)^{\frac{k}{2}}}{c'^{\frac{k}{2}} \sqrt{e^{-h(l_1+\cdots+l_k)}}} \\
&= -\frac{k+2}{4h\pi^2} c' \xi.
\end{aligned}$$

So we have

Theorem 4. *For G is a graph with one vertex and k edges which form k loops, we have*

$$\pi(T, \alpha) = \frac{e^{Th}}{T^{b/2+1}} \left(c_0 + \sum_{n=1}^{N} \frac{c_n(\alpha)}{T^n} + O\left(\frac{1}{T^{N+1}} \right) \right) \text{ as } T \to \infty.$$

The first error term $c_1(\alpha)$ is following.

$$c_1(\alpha) = -\sum_{i=1}^{k} \xi e^{hl_i} \alpha_i^2 + c_{1,0},$$

where $\xi = \frac{1}{2h} (2\pi)^{\frac{k}{2}+2} \sqrt{e^{h(l_1+l_2+\cdots+l_k)}} \left(\sum_{i=1}^k l_i e^{-hl_i} \right)^{\frac{k}{2}+1}$ *and*
$c_{1,0} = \frac{1}{2\pi} \left(d_1 + d_2 - \frac{k+2}{2h \sum_{i=1}^k l_i e^{-hl_i}} \right) \xi$, *$d_1$ and d_2 are specified by (5), (6) and (7).*

Especially, if $k = 2$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ then

$$c_1(\alpha) = -\frac{4\pi^3 (l_1 e^{-hl_1} + l_2 e^{-hl_2})^2 \sqrt{e^{h(l_1+l_2)}}}{h} \left(e^{hl_1} \alpha_1^2 + e^{hl_2} \alpha_2^2 \right) + c_{1,0}.$$

Since h satisfies $e^{-hl_1} + e^{-hl_2} = \frac{1}{2}$, the constant $c_{1,0}$ is

$$\begin{aligned}
c_{1,0} = \; & \frac{4\pi^3 \sqrt{e^{h(l_1+l_2)}}}{96h} \Big\{ [108 + 12(e^{hl_1} + e^{hl_2})](l_1 e^{-hl_1} + l_2 e^{-hl_2})^2 \\
& - 38 l_1 l_2 - 63(l_1^2 e^{-hl_1} + l_2^2 e^{-hl_2}) \Big\}.
\end{aligned}$$

6. Example 2. Let G be a graph with two vertices and three edges which form two loops (Figure 6.1). It can be coded with the following directed graph (Figure 6.2).

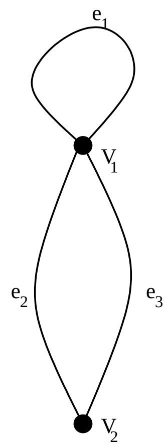

Figure 6.1

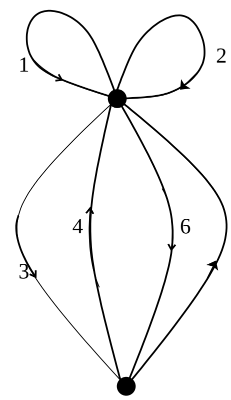

Figure 6.2

The matrix A_G associated with G_o (Figure 6.2) is

$$A_G = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Let the lengths of e_1, e_2, e_3 be l_1, l_2 and l_3, respectively such that conditions (A) and (B) satisfied.

We define

$$r(x) = r(x_0) = \begin{cases} l_1 & \text{if } x_0 = 1 \text{ or } x_0 = 2 \\ l_2 & \text{if } x_0 = 3 \text{ or } x_0 = 4 \\ l_3 & \text{if } x_0 = 5 \text{ or } x_0 = 6. \end{cases}$$

And $f(x) = f(x_0) = (f_1(x_0), f_2(x_0))$ such that

$$f_1(x) = f_1(x_0) = \begin{cases} 1 & \text{if } x_0 = 1 \\ -1 & \text{if } x_0 = 2 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_2(x) = f_2(x_0) = \begin{cases} \frac{1}{2} & \text{if } x_0 = 3 \text{ or } x_0 = 5 \\ -\frac{1}{2} & \text{if } x_0 = 4 \text{ or } x_0 = 6 \\ 0 & \text{otherwise.} \end{cases}$$

In this case, $H_1(G,\mathbb{Z}) = \mathbb{Z}^2$. there exist a measure μ_{-hr} which is Markov measure. That is if we denote by $\mu_{-hr}(x_0, x_1, \ldots, x_n)$ the measure of cylinder $\{x : x = x_0 x_1 \ldots x_n * \cdots\}$ then

$$\begin{aligned}
&(1) \quad \mu_{-hr}(x_0, x_1, \ldots, x_n) \geq 0; \\
&(2) \quad \sum_{x_0} \mu_{-hr}(x_0) = 1; \\
&(3) \quad \mu_{-hr}(x_0, x_1, \ldots, x_n) = \sum_{x_{n+1}} \mu_{-hr}(x_0, x_1, \ldots, x_{n+1}).
\end{aligned}$$

In this case, it is also satisfy following.

$$\mu_{-hr}(1) = \mu_{-hr}(2), \quad \mu_{-hr}(3) = \mu_{-hr}(4), \quad \mu_{-hr}(5) = \mu_{-hr}(6).$$

If we write $n(1) = 2$, $n(2) = 1$, $n(3) = 4$, $n(4) = 3$, $n(5) = 6$ and $n(6) = 5$, by the symmetry of graph G, we have

$$\mu_{-hr}(x_0, x_1, \ldots, x_{n-1}) = \mu_{-hr}(n(x_{n-1}), \ldots, n(x_1), n(x_0)).$$

In order to calculate $\nabla^2\beta(0)$, we will use another expression for $\nabla^2\beta(0)$ in the form,

$$\frac{\partial^2\beta(0)}{\partial u_i \partial u_j} = \frac{1}{\int r d\mu_{-hr}} \lim_{n\to\infty} \frac{1}{n} \int f_i^n f_j^n d\mu_{-hr}.$$

We first prove the following lemma by induction.

Lemma 2. $\forall n \in \mathbb{N}$,

$$\int f_1^n f_2^n d\mu_{-hr} = 0$$

Proof.
(1) Since

$$f_1(x) f_2(x) \equiv 0,$$

by definition of f, lemma holds for $n = 1$.
(2)We assume that lemma holds for $n = k \in \mathbb{N}$, then

$$\int f_1^k f_2^k d\mu_{-hr} = 0.$$

That is

$$\sum_{x_0, x_1, \ldots, x_{k-1}} (f_1(x_0) + \cdots + f_1(x_{k-1}))\,(f_2(x_0) + \cdots + f_2(x_{k-1}))\,\mu_{-hr}(x_0, \ldots, x_{k-1}) = 0.$$

For $n = k+1$,

$$
\begin{aligned}
&\int f_1^{k+1} f_2^{k+1} d\mu_{-hr} = \sum_{x_0,\ldots,x_{k-1},x_k} \left(f_1(x_0) + \cdots + f_1(x_{k-1}) + f_1(x_k)\right) \\
\times\ & \left(f_2(x_0) + \cdots + f_2(x_{k-1}) + f_2(x_k)\right) \mu_{-hr}(x_0,\ldots,x_{k-1},x_k) \\
=\ & \sum_{x_0,\ldots,x_{k-1},x_k} \left(f_1(x_0) + \cdots + f_1(x_{k-1})\right) \left(f_2(x_0) + \cdots + f_2(x_{k-1})\right) \mu_{-hr}(x_0,\ldots,x_{k-1},x_k) \\
+\ & \sum_{x_0,\ldots,x_{k-1},x_k} f_1(x_k) \left(f_2(x_0) + \cdots + f_2(x_{k-1})\right) \mu_{-hr}(x_0,\ldots,x_{k-1},x_k)
\end{aligned}
$$

$$
\begin{aligned}
+\ & \sum_{x_0,\ldots,x_{k-1},x_k} f_2(x_k) \left(f_1(x_0) + \cdots + f_1(x_{k-1})\right) \mu_{-hr}(x_0,\ldots,x_{k-1},x_k) \\
+\ & \sum_{x_0,\ldots,x_{k-1},x_k} f_1(x_k) f_2(x_k) \mu_{-hr}(x_0,\ldots,x_{k-1},x_k).
\end{aligned}
$$

By induction assumption, we have

$$
\begin{aligned}
& \sum_{x_0,\ldots,x_{k-1},x_k} \left(f_1(x_0) + \cdots + f_1(x_{k-1})\right) \left(f_2(x_0) + \cdots + f_2(x_{k-1})\right) \mu_{-hr}(x_0,\ldots,x_{k-1},x_k) \\
=\ & \sum_{x_0,\ldots,x_{k-1}} \left(f_1(x_0) + \cdots + f_1(x_{k-1})\right) \left(f_2(x_0) + \cdots + f_2(x_{k-1})\right) \sum_{x_k} \mu_{-hr}(x_0,\ldots,x_{k-1},x_k) \\
=\ & \sum_{x_0,\ldots,x_{k-1}} \left(f_1(x_0) + \cdots + f_1(x_{k-1})\right) \left(f_2(x_0) + \cdots + f_2(x_{k-1})\right) \mu_{-hr}(x_0,\ldots,x_{k-1}) \\
=\ & \int f_1^k f_2^k d\mu_{-hr} = 0.
\end{aligned}
$$

Since

$$A(x_{k-1}, 1) = 1 \Longleftrightarrow A(x_{k-1}, 2) = 1,$$

and

$$\mu_{-hr}(x_0,\ldots,x_{k-1},1) = \mu_{-hr}(x_0,\ldots,x_{k-1},2),$$

we have

$$
\begin{aligned}
& \sum_{x_0,\ldots,x_{k-1},x_k} f_1(x_k) \left(f_2(x_0) + \cdots + f_2(x_{k-1})\right) \mu_{-hr}(x_0,\ldots,x_{k-1},x_k) \\
=\ & \sum_{x_0,\ldots,x_{k-1},1} f_1(1) \left(f_2(x_0) + \cdots + f_2(x_{k-1})\right) \mu_{-hr}(x_0,\ldots,x_{k-1},1) \\
+\ & \sum_{x_0,\ldots,x_{k-1},2} f_1(2) \left(f_2(x_0) + \cdots + f_2(x_{k-1})\right) \mu_{-hr}(x_0,\ldots,x_{k-1},2) \\
=\ & 0.
\end{aligned}
$$

Similarly,

$$\sum_{x_0,\dots,x_{k-1},x_k} f_2(x_k)\left(f_1(x_0)+\cdots+f_1(x_{k-1})\right)\mu_{-hr}(x_0,\dots,x_{k-1},x_k)=0.$$

It is always true for

$$\sum_{x_0,\dots,x_{k-1},x_k} f_1(x_k)f_2(x_k)\mu_{-hr}(x_0,\dots,x_{k-1},x_k)=0.$$

So

$$\int f_1^{k+1}f_2^{k+1}d\mu_{-hr}=0.$$

(3) Hence

$$\forall n\in\mathbb{N}, \int f_1^n f_2^n d\mu_{-hr}=0.$$

Similarly,

$$\forall n\in\mathbb{N}, \int f_2^n f_1^n d\mu_{-hr}=0.$$

We also need to calculate $\int (f_1^n)^2 d\mu_{-hr}$ and $\int (f_2^n)^2 d\mu_{-hr}$. We have

Lemma 3. $\forall n\in\mathbb{N}$,

$$\int (f_1^n)^2 d\mu_{-hr}=n(\mu_{-hr}(1)+\mu_{-hr}(2))=2n\mu_{-hr}(1).$$

And

$$\int (f_2^n)^2 d\mu_{-hr}=\frac{1}{4}n(\mu_{-hr}(3)+\mu_{-hr}(4)+\mu_{-hr}(5)+\mu_{-hr}(6))=\frac{1}{2}n(\mu_{-hr}(3)+\mu_{-hr}(5)).$$

Proof. We prove this Lemma by induction as we did for Lemma 2.

Hence we have

$$\beta''(\mathbf{0})=\frac{1}{\int r d\mu_{-hr}}\begin{pmatrix} 2\mu_{-hr}(1) & 0 \\ 0 & \frac{1}{2}(\mu_{-hr}(3)+\mu_{-hr}(5)). \end{pmatrix}$$

As we see that in last section,

$$\mu_{-hr}(1)=\mu_{-hr}(2)=e^{-hl_1},$$

and

$$\mu_{-hr}(3)=\mu_{-hr}(4)=e^{-hl_2},$$

and

$$\mu_{-hr}(5)=\mu_{-hr}(6)=e^{-hl_3}.$$

So

$$\beta''(\mathbf{0}) = \frac{1}{2(l_1 e^{-hl_1} + l_2 e^{-hl_2} + l_3 e^{-hl_3})} \begin{pmatrix} 2e^{-hl_1} & 0 \\ 0 & \frac{1}{2}(e^{-hl_2} + e^{-hl_3}). \end{pmatrix}$$

Now we can obtain a_{ij}. Since $\beta''(0)$ is diagonal, we still have $a_{12} = a_{21} = 0$. Let

$$c' = \frac{1}{\int r d\mu_{-hr}} = \frac{1}{2(l_1 e^{-hl_1} + l_2 e^{-hl_2} + l_3 e^{-hl_3})},$$

then

$$\begin{aligned} a_{11} &= \frac{2\pi^2}{h} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}c'(2e^{-hl_1} v_1^2 + \frac{1}{2}(e^{-hl_2} + e^{-hl_3})v_2^2)} v_1^2 dv_1 dv_2 \\ &= \frac{2\pi^2}{h} \frac{\sqrt{2\pi}}{\sqrt{\frac{1}{2}c'(e^{-hl_2} + e^{-hl_3})}} \frac{\sqrt{2\pi}}{2c' e^{-hl_1} \sqrt{2c' e^{-hl_1}}} \\ &= \frac{2\pi^3}{c'^2 h} \frac{e^{hl_1}}{\sqrt{e^{-h(l_1+l_2)} + e^{-h(l_1+l_3)}}} \end{aligned}$$

and

$$\begin{aligned} a_{22} &= \frac{2\pi^2}{h} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}c'(2e^{-hl_1} v_1^2 + \frac{1}{2}(e^{-hl_2} + e^{-hl_3})v_2^2)} v_2^2 dv_1 dv_2 \\ &= \frac{2\pi^2}{h} \frac{\sqrt{2\pi}}{\frac{1}{2}c'(e^{-hl_2} + e^{-hl_3}) \sqrt{\frac{1}{2}c'(e^{-hl_2} + e^{-hl_3})}} \frac{\sqrt{2\pi}}{\sqrt{2c' e^{-hl_1}}} \\ &= \frac{8\pi^3}{c'^2 h} \frac{e^{h(l_2+l_3)}}{(e^{hl_2} + e^{hl_3}) \sqrt{e^{-h(l_1+l_2)} + e^{-h(l_1+l_3)}}}. \end{aligned}$$

Let

$$c = \frac{8\pi^3 (l_1 e^{-hl_1} + l_2 e^{-hl_2} + l_3 e^{-hl_3})^2}{h\sqrt{e^{-h(l_1+l_2)} + e^{-h(l_1+l_3)}}}.$$

We have

Theorem 5. *Let G be a graph with two vertices and three edges which form two loops.*

$$\pi(T, \alpha) = \frac{e^{Th}}{T^{b/2+1}} \left(c_0 + \sum_{n=1}^{N} \frac{c_n(\alpha)}{T^n} + O\left(\frac{1}{T^\delta}\right) \right) \text{ as } T \to \infty$$

with

$$c_1(\alpha) = -c e^{hl_1} \alpha_1^2 - 4c \frac{e^{h(l_2+l_3)}}{e^{hl_2} + e^{hl_3}} \alpha_2^2 + c_{1,0},$$

where

$$c = \frac{8\pi^3 (l_1 e^{-hl_1} + l_2 e^{-hl_2} + l_3 e^{-hl_3})^2}{h\sqrt{e^{-h(l_1+l_2)} + e^{-h(l_1+l_3)}}}.$$

and $c_{1,0}$ is a constant (which , since it is rather complicated, we do not specify here).

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